# Qualifying Exam (May 2022): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10

All problems are weighted equally. On this cover page write which seven problems you want graded.

**problems to be graded:**

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

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**Signature**

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**(1).** Consider the following LP problem and its optimal final tableau shown below:

$$\begin{array}{lrcl} \max\ z & = & 2x_1 + x_2 - x_3 & \\ \text{s.t.} & & x_1 + 2x_2 + x_3 & \leq 8 \\ & & -x_1 + x_2 - 2x_3 & \leq 4 \\ & & x_1\ ,\ x_2\ ,\ x_3 & \geq 0 \end{array}$$

Final Tableau:

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 3 | 2 | 0 | 16 |
| 0 | 1 | 2 | 1 | 1 | 0 | 8 |
| 0 | 0 | 3 | -1 | 1 | 1 | 12 |

**(a)** Write the dual problem and give the optimal dual variable values from the foregoing tableau.

**(b)** Using sensitivity analysis find a new optimal solution if the coefficient of $x_2$ in the objective function is changed from 1 to 3.

**(c)** Suppose that the following constraint is added to the problem:
$x_2 + 2x_3 = 3$. Using sensitivity analysis find the new optimal solution.

**(d)** If you were to choose between increasing the right-hand-side of the first and second constraints, which one would you choose? Why? What is the effect of this increase on the optimal value of the objective function?

**(e)** Suppose that a new activity $x_6$ is proposed with unit return 6 and consumption vector $\mathbf{a}_6 = (2, 1)^t$. Find a new optimal solution. Is the new activity worth considering?

**(2).** Consider the assignment problem shown in the following table (rows: tasks; columns: individuals):

| | Individual | | |
|---|---|---|---|
| Task | 1 | 2 | 3 |
| 1 | 17 | 18 | 16 |
| 2 | 14 | 19 | 17 |
| 3 | 15 | 19 | 18 |

**(a)** Write the linear programming formulation of this problem.

**(b)** Write the dual of the linear program formulated in part (a). List the complementary slackness conditions.

**(c)** Using the Hungarian algorithm, find an optimal primal and an optimal dual solution.

**(d)** Show that the optimal primal and dual solutions found in part (c) satisfy primal feasibility, dual feasibility and complementary slackness.

**(3).**

**(a)** Prove that the system of inequalities $A\mathbf{x} \leq \mathbf{b}$ has a solution $\mathbf{x}$ iff (if and only if) there exists no nonnegative vector $\mathbf{y}$ such that $\mathbf{y}^T A = 0$ with $\mathbf{y}^T \mathbf{b} < 0$.

**(b)** Use part (a) to prove that the system

$$\begin{aligned} -2x_1 + 3x_2 &\leq -2 \\ 3x_1 - x_2 &\leq 2 \\ -11x_1 - x_2 &\leq -7 \end{aligned}$$

has no solution.

**(4).** Consider a street with two lanes. Cars pass through lane $i$ according to a Poisson process with rate $\lambda_i$, $i = 1, 2$. Assume that these two Poisson processes are independent. A woman who wants to cross the street need $a$ seconds to cross lane 1 and $b$ seconds to cross lane 2. Let $T$ be the first time at which the woman could successfully cross the street (both lanes) in one go. Compute $E[T]$.

**(5).** Let $\{X(t), t \geq 0\}$ be a CTMC with stationary distribution $\pi$. Let $\{N(t), t \geq 0\}$ be a Poisson process with intensity $\lambda$ that is independent of $\{X(t)\}$. Denote by $S_n$ the time of the occurrence of the $n$th event in $\{N(t)\}$. Define $Y_n = X(S_n^+)$, i.e., the value of the CTMC immediately after the time of the $n$th event in $\{N(t)\}$. Show that $\{Y_n, n = 0, 1, \ldots\}$ is a discrete-time Markov chain and find its stationary distribution.

**(6).** Consider a single server queue with an FCFS service discipline. Customers arrive according to a Poisson process with rate $\lambda$. The service times are i.i.d. exponentially distribution with parameter $\mu$. An arriving customer who sees $n$ customers in system will only join the system with probability $(n+1)/(n+2)$. (1) State the stability condition for the queueing system. (2) Assuming that the system is stable, compute the long run expected time a customer spends in the waiting line (not including service time).

**(7).** Suppose real-valued $(X_n)_{n \geq 1}$ have distribution functions $F_n$, and that $X_n \xrightarrow{\mathcal{D}} X$. Let $p > 0$ and show that for every positive $N$,

$$\int_{-N}^{N} 1_{\{-N<x<N\}} |x|^p F(dx) \leq \limsup_{n\to\infty} \int_{-N}^{N} 1_{\{-N<x<N\}} |x|^p F_n(dx) < \infty,$$

where $F$ is the distribution of $X$.

**(8).** Let $T$ be a stopping time. For $A \in \mathcal{F}_T$, define

$$T_A(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in A; \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Show that $T_A$ is a stopping time.

**(9).** (a). How efficiently (in big-Oh notation) can the (Euclidean) Delaunay diagram be constructed for $n$ points in the plane? What about for $n$ points in 3D? *Describe briefly (without giving details) one method that achieves the time bound in 2D.*

(b). For $n$ *distinct* points in the plane (in *any* positions, degenerate or not) what is the *fewest* number of edges possible in the Delaunay diagram? Explain. What is the *most* number of edges possible in the Delaunay diagram? Explain.

(c). Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of $n$ points in the plane. Suppose we want only to determine whether or not the pair $(p_1, p_2)$ defines a Delaunay edge. Describe how this can be done efficiently and state the running time.

(d). Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of $n$ points in the plane. Suppose we want only to determine whether or not the point $p_1$ is a vertex of the furthest site Delaunay diagram. How efficiently can this be done? Justify briefly.

(e). When the Fortune sweep algorithm is applied to a set of $n$ points in the plane, what is kept stored in the sweep line status (SLS)? Describe briefly, but clearly.

**(10).** Consider the randomized incremental method for solving a 2-variable linear program. (Assume the objective function is to find the lowest (min-$y$) point in the intersection of the $n$ halfplanes, $\{h_1, \ldots, h_n\}$.)

(a). At certain stages of the algorithm, we are sometimes required to solve a one-dimensional LP. Draw an example of a case in which the addition of the $i$th constraint (halfspace $h_i$) requires the solution of a one-dimensional LP; give also an example of a case in which the addition of $h_i$ does not require the solution of a one-dimensional LP. How does the algorithm tell which case occurs? (i.e., what test is done on each $h_i$?)

(b). Describe briefly *what* a "one-dimensional LP" problem is, how it is defined in terms of the added constraint $h_i$, and how it is solved. What is the running time of the solution method? (in big-Oh notation)

(c). What is the probability that, during the solution of a 2-variable LP, we are required to solve a one-dimensional LP upon insertion of halfplane (constraint) $h_i$ (the $i$th constraint inserted during the algorithm)? Why? (give a brief justification)

(d). We have seen various randomized incremental algorithms, in addition to 2-variable LP. State one such result. You need not describe the algorithm, just state what it solves, and what its expected running time is.